Definition of a Limit Solutions

Here are the solutions to the Definition of a Limit exercises.

Question 1

For each of the following limits, find $\delta$ in terms of $\epsilon$.

(a) $\lim_{x \to 2} x + 2 = 4$

We wish to keep $|x + 2 - 4| < \epsilon$ for any $\epsilon > 0$, by keeping $|x - 2| < \delta$. It is easy to simplify $|x + 2 - 4| < \epsilon$ to $|x - 2| < \epsilon$. So, we see in this case that $\delta = \epsilon$.

(b) $\lim_{x \to 5} 2x - 3 = 7$

We wish to keep $|2x - 3 - 7| < \epsilon$ for any $\epsilon > 0$, by keeping $|x - 5| < \delta$. It is easy to do this as follows:

$$|2x - 3 - 7| < \epsilon$$

$$|2x - 10| < \epsilon$$

$$2|x - 5| < \epsilon$$

$$|x - 5| < \epsilon/2$$

So, we see in this case that $\delta = \epsilon/2$.

(c) $\lim_{x \to 2} 4x + 3 = 11$

We wish to keep $|4x + 3 - 11| < \epsilon$ for any $\epsilon > 0$, by keeping $|x - 2| < \delta$. It is easy to do this as follows:

$$|4x + 3 - 11| < \epsilon$$

$$|4x - 8| < \epsilon$$
\[4|x - 2| < \epsilon\]

\[|x - 2| < \epsilon/4\]

So, we see in this case that \(\delta = \epsilon/4\).

(d) \(\lim_{x \to 1} 6 - 2x = 4\)

We wish to keep \(|6 - 2x - 4| < \epsilon\) for any \(\epsilon > 0\), by keeping \(|x - 1| < \delta\). It is easy to do this as follows:

\[|6 - 2x - 4| < \epsilon\]

\[|2 - 2x| < \epsilon\]

\[|2x - 2| < \epsilon\]

\[2|x - 1| < \epsilon\]

\[|x - 1| < \epsilon/2\]

So, we see in this case that \(\delta = \epsilon/2\).

(e) \(\lim_{x \to 6} x^2 - 30 = 6\)

We wish to keep \(|x^2 - 30 - 6| < \epsilon\) for any \(\epsilon > 0\), by keeping \(|x - 6| < \delta\). This is harder than the linear case.

\[|x^2 - 30 - 6| < \epsilon\]

\[|x^2 - 36| < \epsilon\]

\[|(x - 6)(x + 6)| < \epsilon\]

\[|x - 6|.|x + 6| < \epsilon\]
\[|x - 6| < \frac{\epsilon}{|x + 6|}\]

Now \(|x + 6|\) is going to be pretty close to 12, so to be safe we can choose \(\delta = \epsilon/13\).

\[(f)\ \lim_{x \to 2} \frac{1}{x} = \frac{1}{2}\]

We wish to keep \(|1/x - 1/2| < \epsilon\) for any \(\epsilon > 0\), by keeping \(|x - 2| < \delta\).

\[|1/x - 1/2| < \epsilon\]

\[|\frac{2 - x}{2x}| < \epsilon\]

\[\frac{|x - 2|}{|2x|} < \epsilon\]

\[|x - 2| < \epsilon|2x|\]

Now \(2x\) is pretty close to 4, so to be safe we choose \(\delta = 3\epsilon\).

\[(g)\ \lim_{x \to 9} \frac{1}{(x - 3)} = \frac{1}{6}\]

We wish to keep \(|1/(x - 3) - 1/6| < \epsilon\) for any \(\epsilon > 0\), by keeping \(|x - 9| < \delta\).

\[|1/(x - 3) - 1/6| < \epsilon\]

\[|\frac{6 - x + 3}{6(x - 3)}| < \epsilon\]

\[|\frac{9 - x}{6(x - 3)}| < \epsilon\]

\[|x - 9| < \epsilon|6(x - 3)|\]

Now \(6(x - 3)\) is pretty close to 36, so to be safe we choose \(\delta = 35\epsilon\).

\[(h)\ \lim_{x \to 2} \frac{1}{(2x + 1)} = \frac{1}{5}\]

We wish to keep \(|1/(2x + 1) - 1/5| < \epsilon\) for any \(\epsilon > 0\), by keeping \(|x - 2| < \delta\).

\[|1/(2x + 1) - 1/5| < \epsilon\]
\[
\left| \frac{5-2x-1}{5(2x+1)} \right| < \epsilon
\]
\[
\left| \frac{4-2x}{5(2x+1)} \right| < \epsilon
\]
\[
\left| \frac{2}{5(2x+1)} \right| |x-2| < \epsilon
\]
\[
|x-2| < \epsilon |5(2x+1)|/2
\]

Now \( |5(2x+1)|/2 \) is pretty close to 12.5, so to be safe we choose \( \delta = 12 \epsilon \).

**Question 2**

Use the definition of a limit to **prove** each of the above limits.

(a) \( \lim_{x \to 2} x + 2 = 4 \)

If we choose \( \delta = \epsilon \) and assume \( |x - 2| < \delta = \epsilon \), then
\[
|x + 2 - 4| = |x - 2| < \delta = \epsilon
\]
and we are done.

(b) \( \lim_{x \to 5} 2x - 3 = 7 \)

If we choose \( \delta = \epsilon/2 \) and assume \( |x - 5| < \delta = \epsilon/2 \), then
\[
|2x - 3 - 7| = |2x - 10| = 2|x - 5| < 2\delta = 2\epsilon/2 = \epsilon
\]
and we are done.

(c) \( \lim_{x \to 2} 4x + 3 = 11 \)

If we choose \( \delta = \epsilon/4 \) and assume \( |x - 2| < \delta = \epsilon/4 \), then
\[
|4x + 3 - 11| = |4x - 8| = 4|x - 2| < 4\delta = 4\epsilon/4 = \epsilon
\]
and we are done.
(d) \( \lim_{x \to 6} 6 - 2x = 4 \)
If we choose \( \delta = \epsilon/2 \) and assume \( |x - 1| < \delta = \epsilon/2 \), then

\[
6 - 2x - 4 = |2 - 2x| = 2|x - 1| < 2\delta = 2\epsilon/2 = \epsilon
\]
and we are done.

(e) \( \lim_{x \to 6} x^2 - 30 = 6 \)
If we choose \( \delta = \epsilon/13 \) and assume \( |x - 6| < \delta = \epsilon/13 \), then

\[
|x^2 - 30 - 6| = |x^2 - 36| = |x - 6||x + 6| < \delta|x + 6| = \epsilon \frac{|x + 6|}{13} < \epsilon
\]
provided we are close enough such that \( |x + 6| < 13 \). This is reasonable, as we work under the assumption that \( \epsilon \) (and thus \( \delta \)) will be small.

(f) \( \lim_{x \to 2} 1/x = 1/2 \)
If we choose \( \delta = 3\epsilon \) and assume \( |x - 2| < \delta = 3\epsilon \), then

\[
|1/x - 1/2| = \left| \frac{2 - x}{2x} \right| = \frac{|x - 2|}{|2x|} < \frac{\delta}{|2x|} = \frac{3\epsilon}{|2x|} < \epsilon
\]
provided \( x \) is close enough to 2 such that \( 3 < |2x| \).

(g) \( \lim_{x \to 9} 1/(x - 3) = 1/6 \)
If we choose \( \delta = 35\epsilon \) and assume \( |x - 9| < \delta = 35\epsilon \), then

\[
|1/(x - 3) - 1/6| = \left| \frac{6 - x + 3}{6(x - 3)} \right| = \frac{|x - 9|}{|6(x - 3)|} < \frac{\delta}{|6(x - 3)|} = \frac{35\epsilon}{|6(x - 3)|} < \epsilon
\]
provided \( x \) is close enough to 9 such that \( 35 < 6|x - 3| \).

(h) \( \lim_{x \to 2} 1/(2x + 1) = 1/5 \)
If we choose \( \delta = 12\epsilon \) and assume \( |x - 2| < \delta = 12\epsilon \), then

\[
|1/(2x+1) - 1/5| = \left| \frac{5 - 2x - 1}{5(2x + 1)} \right| = \frac{|4 - 2x|}{|5(2x + 1)|} = \frac{2|x - 2|}{|5(2x + 1)|} < \frac{2\delta}{|5(2x + 1)|} = \frac{12\epsilon}{|5(2x + 1)|} < \epsilon
\]
provided \( x \) is close enough to 2 such that \( 24 < 5|2x + 1| \).
Question 3

Use the definition of a limit as $x \to \pm\infty$ to prove each of the following limits.

(a) $\lim_{x\to\infty} \frac{1}{x} + 2 = 2$

We need to identify a good candidate for $N$ in terms of $\epsilon$, and then show that restricting $x > N$ will force $|\frac{1}{x} + 2 - 2| < \epsilon$. We work backwards as usual:

$|\frac{1}{x} + 2 - 2| < \epsilon$

$|\frac{1}{x}| < \epsilon$

$\frac{1}{x} < \epsilon$

$1/\epsilon < x$

So let’s choose $N = 1/\epsilon$. If we assume that $x > N = 1/\epsilon$ we have

$|\frac{1}{x} + 2 - 2| = |\frac{1}{x}| = 1/x < 1/N = 1/(1/\epsilon) = \epsilon$

and so we are done.

(b) $\lim_{x\to\infty} \frac{1}{3 + x} = 0$

We need to identify a good candidate for $N$ in terms of $\epsilon$, and then show that restricting $x > N$ will force $|\frac{1}{3 + x} - 0| < \epsilon$. We work backwards as usual:

$|\frac{1}{3 + x} - 0| < \epsilon$

$|\frac{1}{3 + x}| < \epsilon$

$\frac{1}{3 + x} < \epsilon$

$1/\epsilon < 3 + x$

$1/\epsilon - 3 < x$
So let’s choose $N = 1/\epsilon - 3$. If we assume that $x > N = 1/\epsilon - 3$ we have

$$|1/(3 + x) - 0| = |1/(3 + x)| = 1/(3 + x) < 1/(3 + N) = 1/(3 + 1/\epsilon - 3) = 1/(1/\epsilon) = \epsilon$$

and so we are done.

(c) $\lim_{x \to \infty} (3x + 2)/(3x + 4) = 1$

We need to identify a good candidate for $N$ in terms of $\epsilon$, and then show that restricting $x > N$ will force $|(3x + 2)/(3x + 4) - 1| < \epsilon$. We work backwards as usual:

$$|(3x + 2)/(3x + 4) - 1| < \epsilon$$

$$|(3x + 2)/(3x + 4) - (3x + 4)/(3x + 4)| < \epsilon$$

$$|(-2)/(3x + 4)| < \epsilon$$

$$2/(3x + 4) < \epsilon$$

$$2/\epsilon < 3x + 4$$

$$\frac{2/\epsilon - 4}{3} < x$$

So let’s choose $N = \frac{2/\epsilon - 4}{3}$. If we assume that $x > N = \frac{2/\epsilon - 4}{3}$ we have

$$|(3x+2)/(3x+4)-1| = |(-2)/(3x+4)| = 2/(3x+4) < 2/(3N+4) = 2/(3\cdot\frac{2/\epsilon - 4}{3} + 4) = 2/(2/\epsilon) = \epsilon$$

and so we are done.

(d) $\lim_{x \to -\infty} 1/(x^3) = 0$

We need to identify a good candidate for $N$ in terms of $\epsilon$, and then show that restricting $x < N$ will force $|1/x^3 - 0| < \epsilon$. We work backwards as usual:

$$|1/x^3 - 0| < \epsilon$$
\[ |1/x^3| < \epsilon \]

\[-1/x^3 < \epsilon \]

\[-1/\epsilon > x^3 \]

\[ \sqrt[3]{-1/\epsilon} > x \]

So let’s choose \( N = \sqrt[3]{-1/\epsilon} \). If we assume that \( x < N = \sqrt[3]{-1/\epsilon} \) we have

\[ |1/x^3 - 0| = |1/x^3| = -1/x^3 < -1/N^3 = -1/(-1/\epsilon) = \epsilon \]

and we are done.

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